INTRO TO TORIC VARIETIES

CONNOR SIMPSON

Warning: these notes were written for my own use and may contain errors, omissions, or slight lies.

Notation for this talk is taken mainly from Fulton. Fulton's *Introduction to toric varieties* and Cox, Little, and Schenk's book are both good references. For simplicity, we stick with the lattice $\mathbb{Z}^d \subset \mathbb{R}^d$ and use affine examples where possible.

1. Motivation

Definition 1.1. A **toric variety** is a normal variety X that contains a torus $T = (\mathbb{C}^*)^d$ as an open dense subset, along with an action $T \times X \to X$ of T on X that extends the action of T on itself.

We like toric varieties because they are

- Built out of simple parts (tori)
- The parts fit together in simple ways

Hence, it is easy to compute many things about them!

A lot of our favorite varieties are easy to realize as toric varieties.

Example 1.2. $(\mathbb{C}^*)^d$ ♢

Example 1.3. $\mathbb{A}^d$. Here, $(\mathbb{C}^*)^d$ is embedded in the usual way as the set of elements with no zero coordintes and acts on $\mathbb{A}^d$ as multiplication by a diagonal matrix. ♢

Example 1.4. $\mathbb{P}^d$. We have

$$(\mathbb{C}^*)^d \hookrightarrow \mathbb{A}^d \hookrightarrow \mathbb{P}^d$$
$$(x_1, \dots, x_d) \mapsto [1 : x_1 : x_2 : \cdots : x_d]$$

. The action extends that on $\mathbb{A}^d$. For $t = (t_1, \dots, t_d) \in (\mathbb{C}^*)^d$, and $x = [x_0 : \cdots : x_d] \in \mathbb{P}^d$.

$$t \cdot x = [x_0 : t_1 x_1 : t_2 x_2 : \cdots : t_d x_d]$$

♢

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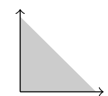
2. The kit of parts: affine toric varieties

Definition 2.1. A **rational cone** in $\mathbb{R}^d$ is $\{\sum_i a_i v_i : \forall i, a_i \geq 0\}$ with $\{v_i\}$ a finite set of vectors with rational coordinates.

Call a cone **strongly convex** if it contains no subspace.

Example 2.2. The first orthant is generated by $\{e_1, e_2\}$.



♢

Example 2.3. The upper half-plane is a cone, but is not strongly convex. ♢

2.1. Cone to variety.

(i) Start with a cone σ
(ii) Let $\sigma^\vee = \{u \in (\mathbb{R}^d)^\vee : \forall v \in \sigma,\, u(v) \geq 0\}$.
(iii) Let $M = \operatorname{Hom}(\mathbb{Z}^n, \mathbb{Z}) \subset (\mathbb{R}^d)^\vee$ and set $S_\sigma := M \cap \sigma^\vee$.
(iv) Let $U_\sigma = \operatorname{Spec} \mathbb{C}[S_\sigma]$. You've built a toric variety!

It's not obvious that this is toric.

Example 2.4 (Building $\mathbb{A}^2$). Take the first orthant from before. Then $\sigma^\vee = \operatorname{cone}(e_1^*, e_2^*)$, and

$$S_\sigma = M \cap \sigma^\vee = \{(a, b) : a, b \in \mathbb{Z}_{\geq 0}\}.$$

To see that $U_\sigma := \operatorname{Spec} \mathbb{C}[S_\sigma] = \mathbb{A}^2$ present by

$$\mathbb{C}[x, y] \to \mathbb{C}[S_\sigma]$$
$$x \mapsto e_1^*$$
$$y \mapsto e_2^*$$

♢

2.2. Principal opens.

Notice that a cone contains other cones: its faces.

Definition 2.5. A **face** of a cone σ is $\sigma \cap u^\perp$ for some $u \in \sigma^\vee$.

To the face $\tau = \sigma \cap u^\perp$, associate the open subvariety

$$U_\tau = \{x \in X : u(x) \neq 0\} = \operatorname{Spec} \mathbb{C}[S_\sigma][u^{-1}] \subset X$$

Example 2.6. Consider the face $\tau = \operatorname{cone}(e_1) \subset \operatorname{cone}(e_1, e_2)$. This is cut out by $e_2^* = 0$. Using our presentation from before, it follows that

$$U_\tau \cong \operatorname{Spec} \mathbb{C}[x, y, y^{-1}]$$

which is the affine on which $y \neq 0$. ♢

Example 2.7. Set $\tau = (0, 0)$. Then τ is cut out by $e_2^* + e_1^*$, so

$$U_\tau \cong \operatorname{Spec} \mathbb{C}[x, y, (xy)^{-1}] \cong (\mathbb{C}^*)^2$$

We found the torus! ♢

In general, the point of the cone will give us the torus embedding.

2.3. Torus action.

- Points of $\operatorname{Spec} \mathbb{C}[S_\sigma]$ are semigroup maps $S_\sigma \to \mathbb{C}^*$.
- The torus acts by $(t \cdot x)(u) = t(u)x(u)$.

Example 2.8. $(a, b) \in \mathbb{A}^2$ corresponds to $x : S_\sigma \to \mathbb{C}^*$ given by $e_1^* \mapsto a$ and $e_2^* \mapsto b$. If $t = (t_1, t_2)$ is a torus element, then $(t \cdot x)(e_1^*) = t(e_1^*)x(e_1^*) = t_1 a$ and likewise for the second coordinate. This is the natural action from the start! ♢

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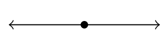
3. Putting the pieces together

Definition 3.1. A **fan** is a set of cones Δ such that for all $\sigma, \sigma' \in \Delta$, $\sigma \cap \sigma' \in \Delta$ and $\sigma \cap \sigma'$ is a face of both σ and σ'.

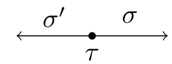
Example 3.2. The following is a fan consisting of two 1-dimensional cones and one zero-dimensional cone



♢

If τ is a face of two cones σ and σ', then we can glue U_σ and $U_{\sigma'}$ together along U_τ.

Example 3.3. For the fan



we have:

$$\mathbb{C}[S_\sigma] \cong \mathbb{C}[x]$$
$$\mathbb{C}[S_{\sigma'}] \cong \mathbb{C}[y]$$
$$\mathbb{C}[S_\tau] \cong \mathbb{C}[z, z^{-1}]$$

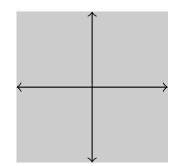
and

$$\mathbb{C}[S_\sigma] \to \mathbb{C}[S_\tau]$$
$$x \mapsto z$$
$$\mathbb{C}[S_{\sigma'}] \to \mathbb{C}[S_\tau]$$
$$y \mapsto z^{-1}$$

so this fan corresponds to $\mathbb{P}^1$. ♢

Proposition 3.4. $X(\Delta \times \Delta') \cong X(\Delta) \times X(\Delta')$.

Example 3.5. The following is the fan for $\mathbb{P}^1 \times \mathbb{P}^1$.



♢

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4. Some stuff you can read off from the fan

4.1. Smoothness.

Example 4.1. If $\sigma = \operatorname{cone}(e_1, 2e_1 + e_2)$, then $\sigma^\vee = \operatorname{cone}(e_1^*, e_1^* + 2e_2^*)$. Hence, $U_\sigma \cong \mathbb{C}[x, y, z] / \langle xz - y^2 \rangle$ which is singular :(♢

Difference between this and previous examples is that the cone generators were not a basis for $\mathbb{Z}^d$.

Proposition 4.2. *U_σ is smooth if and only if σ is generated by a basis for $\mathbb{Z}^d$.*

4.2. Euler characteristic.

Proposition 4.3. *If X is a d-dimensional toric variety, then $\chi(X)$ is the number of d-dimensional cones.*

Example 4.4. The fan for $\mathbb{P}^1$ has two 1-dimensional cones, giving an Euler characteristic of 2. $\diamondsuit$

Example 4.5. The fan for $\mathbb{A}^2$ has one 2-dimensional cone, which gives Euler characteristic of 1. $\diamondsuit$

Example 4.6. The fan for $\mathbb{C}^*$ is one point in $\mathbb{R}^1$, so we get Euler characteristic 0. $\diamondsuit$

More generally, there are fairly simple formulas for all Betti numbers in terms of the number of cones of each dimension.

4.3. Decomposition into orbits.

The action of T on X partitions X into disjoint orbits.

Proposition 4.7. *The orbits are of the form $T \cdot x_\sigma$ for $\sigma \in \Delta$ where*

$$x_\sigma : u \mapsto \begin{cases} 1, & u \in \sigma^\perp \\ 0, & \textit{else} \end{cases}$$

Example 4.8. In $\mathbb{A}^2$, let $\sigma = \text{cone}(e_1, e_2)$, $\tau_1 = \text{cone}(e_1)$, $\tau_2 = \text{cone}(e_2)$. Then

$$\begin{aligned} x_\sigma &: u \mapsto 0 && (0,0) \\ x_{\tau_i} &: u \mapsto \begin{cases} 1, & u = e_j^*, \, j \neq i \\ 0, & \text{else} \end{cases} && (1,0), (0,1) \\ x_{(0,0)} &: e_i^* \mapsto 1 && (1,1) \end{aligned}$$

It is evident that the orbits of these points give all of $\mathbb{A}^2$. Note that fixed points correspond to full dimensional cones. $\diamondsuit$

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5. Blow-ups

5.1. Torus-invariant subvarieties.

The closures of the torus orbits give the irreducible torus-invariant subvarieties $V(\tau)$ for each cone $\tau \in \Delta$. In particular, rays correspond to divisors. The orbit-closure $V(\tau)$ is the complement of the opens defined by the facets of τ.

Example 5.1. In $\mathbb{A}^2$, the orbit-closures are: $\{(0,0)\}$, $\mathbb{C}e_1$, $\mathbb{C}e_2$, and $\mathbb{A}^2$. $\diamondsuit$

5.2. Blow-ups.

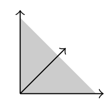
Blowing up along a torus-invariant subvariety gives a toric map $B \to X(\Delta)$, where B is the blow-up. The algorithm to produce the fan of the blow-up of a smooth toric variety along $V(\sigma)$ is

(i) Let $\rho_1, \ldots, \rho_r$ be the rays of σ.
(ii) Let $\rho_0 := \sum_i \rho_i$.
(iii) Form a fan Δ' from Δ by replacing each $\tau \supset \sigma$, with $\tau = \text{cone}(\rho_1, \ldots, \rho_r, \rho_{r+1}, \ldots, \rho_s\}$, with new cones

$$\{\text{cone}(A) : A \subset \{\rho_1, \ldots \rho_{r+1}\}, A \not\supset \{\rho_1, \ldots, \rho_r\}\}$$

You can do more general blow-ups (of non-smooth things). Construction is very similar, slightly more complicated.

Example 5.2. Blowing up $\mathbb{A}^2$ at the origin. The origin corresponds to the single full-dimensional cone of $\mathbb{A}^2$. Set $e_0 = e_1 + e_2$. The new fan is



$\diamondsuit$

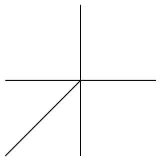
Fuzzily, you can maybe see that this is the blow-up because we added a new ray (exceptional divisor) and the star of that ray is the fan of $\mathbb{P}^1$, which is what we expect.

Concretely, the blowup B is $V(xt_0 - yt_1) \subset \mathbb{A}^2 \times \mathbb{P}^1$. In the $t_0 \neq 0$ chart, this is

$$\mathbb{C}\left[x, y, \frac{t_1}{t_0}\right] / \left\langle x\frac{t_1}{t_0} - y \right\rangle \cong \mathbb{C}[x, x^{-1}y]$$

and $\mathbb{C}[x, x^{-1}y]$ is the semigroup algebra of $\sigma^\vee$, where $\sigma = \text{cone}(e_2, e_1 + e_2)$. The chart where $t_1 \neq 0$ is similarly isomorphic to the open of the other maximal cone.

Example 5.3. Last time, Dima told us that if you blow up $\mathbb{P}^2$ at two points and $\mathbb{P}^1$ at 1 point, you get the same thing. This is very obvious from the toric perspective: the fan below represents the common blow-up:



$\diamondsuit$